

# LIMIT CYCLE WALKING ON A REGULARIZED GROUND

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**ABSTRACT.** The singular nature of contact problems, such as walking, makes them difficult to analyze mathematically. In this paper we will “regularize” the contact problem of walking by approximating the ground with a smooth repulsive potential energy and a smooth dissipative friction force. Using this model we are able to prove the existence of a limit cycle for a periodically perturbed system which consists of three masses connected by springs. In particular, this limit cycle exists in a symmetry reduced phase. In the unreduced phase space, the motion of the masses resembles walking.

## 1. INTRODUCTION

Consider the physics of walking. The number of degrees of freedom in the system seem to change, depending on whether one foot is on the ground or in the air. The on/off nature of contact makes the analysis of such a system difficult. In this paper, we will “regularize” the contact problem by using smooth viscous friction and potential energies to model the ground, resulting in a dissipative system on a phase space  $TQ$ . We will then identify an  $\mathbb{R}$  symmetry of the system and find a reduced order model on the quotient space  $TQ/\mathbb{R}$ . Upon finding a hyperbolically stable fixed point in  $TQ/\mathbb{R}$ , we will create a limit cycle by adding a periodic perturbation. This limit cycle in  $TQ/\mathbb{R}$  corresponds to a “relative limit cycle,” or *walking*, in the original phase space  $TQ$  (see figure 1).

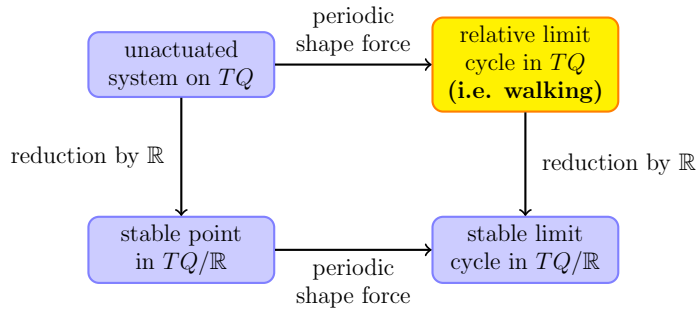


FIGURE 1. This commutative diagram characterizes walking as a relative limit cycle in a dissipative Lagrangian system.

**1.1. Main Contributions.** The main contributions of this paper are:

- (1) We present a model of a walker in  $\mathbb{R}^2$  where the no-slip condition and the no penetration condition have been “smeared” over a region around  $y = 0$ . This model will be a dissipative Lagrangian system on a tangent bundle  $TQ$  where  $Q = \mathbb{R}^6$ .
- (2) We will determine an  $\mathbb{R}$ -symmetry for the system which corresponds to translation along the  $x$ -axis. We will also state the reduced equations of motion on  $TQ/\mathbb{R}$ .
- (3) We prove the existence of a stable point for our 2d walker in  $TQ/\mathbb{R}$ .
- (4) We will periodically perturb the system by oscillating the equilibrium length of the legs. This will allow us to assert the existence of a limit cycle in  $TQ/\mathbb{R}$ .
- (5) The limit cycle in  $TQ/\mathbb{R}$  will be related to trajectories in  $TQ$  via a phase reconstruction formula. We will find that when the phase is non-zero, the trajectories in  $TQ$  correspond to “walking” as defined in this paper.

**1.2. Background & Motivation.** This “regularized” approach is in contrast to the “hybrid systems” approach, where transitions between different types of phase spaces are given by various transition maps. The hybrid systems approach is arguably more accurate, and has yielded a number of insights and useful models. For example, a hybrid systems formulation was introduced in [12] where the transition maps led to a dimension reduction; it was suggested that a limit cycle was approached passively. Since the work of [12], the notion of walking as a limit cycle has become more common, and more sophisticated analysis have lent further support to this idea [5] [6]. The most compelling arguments are the original videos of McGreer which accompany [12].

On the biological side, the notion of “central pattern generators” (CPGs) has been hypothesized to be a fundamental neural mechanism used in walking [7]. These CPGs are non-localized collections of neurons which produce rhythmic activity, and respond to various inputs which modulate these rhythms. Therefore the link between CPGs and limit cycle walking is one which links periodic activation of the controls to periodic motion of the body. This link is used in the creation of simple models which can be feasibly analyzed (see for example [6]).

Lastly, viewing walking as a limit cycle allows for great reductions in complexity. In particular, under weak assumptions, the existence of limit cycles in hybrid systems implies the existence of a reduced order model for the system as a whole [2]. The most recent paper, [2], dealt with the singular nature of hybrid systems by using relatively weak assumptions on the transition maps to “smooth” the dynamics across the transition regions and prove the existence of a lower-order hybrid model. Our paper can be seen as a “dual approach” to [2] in that we regularize the transitions maps themselves. To do this we will use viscous friction force to approximate the no-slip condition as described in [11] and §1.6 of [1].

Using our smooth system we will model walking as a limit cycle. It seems fair to ask “in what space does this limit cycle exist”. In this paper, we will be concerned with dynamics on a phase space,  $TQ$ , and the dynamics will be those of a forced Lagrangian system. We will find that without active forcing, the system is dissipative and symmetric with respect to translations. This will allow us to perform Lagrangian reduction [3] to obtain an asymptotically stable point in  $TQ/\mathbb{R}$ . By perturbing this stable point we will be able to assert the existence of a limit cycle in  $TQ/\mathbb{R}$  using the “persistence theorem” of [4] and [9]. This construction can be seen as a terrestrial counterpart to recent work done by the author on swimming wherein the phase space for fluid structure interaction,  $\mathcal{A}$ , exhibits an  $SE(3)$  symmetry which allows us to interpret swimming as a limit cycle on  $\mathcal{A}/SE(3)$  [10].

**1.3. Acknowledgements.** The notion of realizing the no-slip condition as a limit of viscous friction was brought to my attention by Dmitry V. Zenkov in a conversation about the long time behavior of such systems. I was educated by Sam Burden on the role of limit cycles in model reduction for hybrid systems. A conversation which involved banging our fists on tables over lunch convinced me that the role of phase space contraction was important for the passive control of these systems. Simultaneously Erica J. Kim asked how to view general locomotion as a limit cycle, and spurred my own pursuit on a project for viewing swimming as a limit cycle. The present paper can be seen as a terrestrial counterpart to this. I would like to thank Jaap Eldering for pointing out flaws in an earlier formulation of the model and providing much encouragement and interest in the project. Finally, the initial stimulus to write this paper was given by Jair Koiller, who has been very supportive of my recent forays into biomechanics. This research was supported by the European Research Council Advanced Grant 267382 FCCA and NSF grant CCF-1011944.

## 2. THE MODEL

The model can be broken into two distinct components: the walker and the environment. We will discuss the model of the walker in a vacuum before we elaborate on how to model the ground, gravity, and the no-slip condition.

**2.1. A model of a walker (in a vacuum).** The walker consists of three point particles of unit mass all connected by springs of unit stiffness with light viscous damping. These point particles move through  $\mathbb{R}^2$  with positions  $q_i = (x_i, y_i)$  and velocities  $\dot{q} = (\dot{x}_i, \dot{y}_i)$  for  $i = 1, 2, 3$ . If we let  $Q = \mathbb{R}^6$  with coordinates  $q = (q_1, q_2, q_3) \equiv (x_1, y_1, x_2, y_2, x_3, y_3)$ , we can describe the walker in a vacuum as a Lagrange-mechanical system. Here kinetic energy is  $K(q, \dot{q}) = \frac{1}{2}\|\dot{q}\|^2$  and potential

energy from the springs is

$$U_{ij}(q_i, q_j) = \frac{1}{2} \left( \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} - \ell_{ij}(t) \right)^2,$$

where  $U_{ij}(q_i, q_j)$  is the potential energy stored in the spring which connects mass  $i$  with mass  $j$  for some time dependent length  $\ell_{ij}(t)$ . The functions  $\ell_{ij} = \ell_{ij}(t)$  do not control the shape of the walker directly, but instead determine an energy-minimizing shape and therefore control it indirectly.

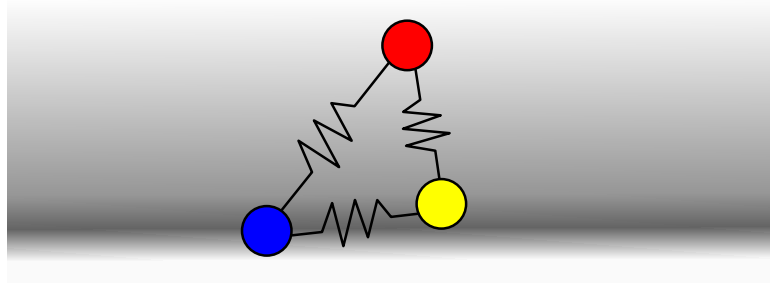


FIGURE 2. Depicted is a cartoon of our walker. The grey cloud in the background represents the potential energy due to gravity plus the potential energy from the ground,  $U_{np}$  of equation (2.2).

The Lagrangian is given by

$$L(q, \dot{q}; t) = \frac{1}{2} \|\dot{q}\|^2 - \sum_{i < j} U_{ij}(q_i, q_j)$$

and we define the viscous force on the spring between particle  $i$  and  $j$  by

$$F_{ij} = (\dot{x}_j - \dot{x}_i)dx_i + (\dot{y}_j - \dot{y}_i)dy_i.$$

The equations of motion are  $\ddot{q} = \left( \sum_{i,j=1}^3 F_{ij} \right) - \left( \sum_{i < j} dU_{ij} \right)$ , and explicitly given in terms of  $q_i$  and  $q_j$  as

$$\ddot{q}_i = \sum_{i \neq j} ((\|q_j - q_i\| - \ell_{ij}(t)) \hat{\mathbf{n}}_{ij} + \dot{q}_j - \dot{q}_i)$$

where  $q_i = (x_i, y_i)$  and  $\hat{\mathbf{n}}_{ij}$  is the unit vector which points from mass  $i$  to mass  $j$ . As this is ill-defined when  $q_i = q_j$ , the equations are ill-defined on a set of measure zero.

**2.2. A regularized model of the ground.** Now we will present a model for the ground on which the walker will walk. The surface of the ground occupies

the line  $\{y = 0\}$  in  $\mathbb{R}^2$ . Ideally, the ground is unpenetrable and imposes a no-slip condition. This is mathematically represented by the constraints

$$\begin{aligned} (1) \quad & y_i \geq 0 \\ (2) \quad & \dot{x}_i = 0 \text{ if } y_i = 0 \end{aligned}$$

for  $i = 1, 2, 3$  where equation (1) is the *no-penetration condition* and equation (2) is the *no-slip condition*. Both the no-penetration and no-slip conditions present challenges of a singular nature as they are conditions which abruptly “turn on” at  $y = 0$  and are inactive otherwise. It is precisely this “on/off” character which we will regularize.

Firstly, the no-penetration condition can be realized by taking a potential energy that is infinite for  $y < 0$  and finite for  $y \geq 0$ . We will model this non-smooth potential by approximating it with a smooth one. In particular, the hyperbolic tangent

$$\tanh(x) = \frac{e^{2x} - 1}{e^{-2x} + 1}$$

serves as a smooth approximation of a step function and allows us to define the potential energy  $U_{\text{np}} : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$U_{\text{np}}(q_i) = -\mu_{\text{np}} \tanh\left(\frac{y_i}{\sigma_{\text{np}}}\right),$$

which has the gradient

$$dU_{\text{np}}(q_i) = \frac{\mu_{\text{np}}}{\sigma_{\text{np}}} \left[ \left( \tanh\left(\frac{y_i}{\sigma_{\text{np}}}\right) \right)^2 - 1 \right] dy_i$$

and prevents the  $i$ th particle from falling through the floor. The parameter  $\sigma_{\text{np}}$  controls the width of the region in which this force is active while  $\mu_{\text{np}}$  amplifies this force. As  $\sigma_{\text{np}}$  approaches 0, this force becomes concentrated around the line  $\{y = 0\}$  and becomes infinitely strong; thus our model approaches an exact model when  $\sigma_{\text{np}}$  approaches zero and  $\mu_{\text{np}}$  remains constant.

The no-slip condition is similar to the no-penetration condition in that it is only active at  $\{y = 0\}$ . However, unlike the no-penetration condition, the no-slip condition is not derivable from a potential energy but instead can be viewed as a limit of viscous friction [11]. In particular, consider the viscous force on the  $i$ th particle given by

$$F_{\text{ns}}(q_i, \dot{q}_i) = -\mu_{\text{ns}} \dot{x}_i \left[ \frac{1}{2} \tanh\left(\frac{y_i}{\sigma_{\text{ns}}}\right) - \frac{1}{2} \right] dx_i.$$

This force “turns on” in the region near  $\{y \leq 0\}$  and dampens motion of the particles in that area. In other words,  $F_{\text{ns}}$  resists slipping but does not outlaw it. As before, the parameter  $\sigma_{\text{ns}}$  controls the width of the boundary where this force

is active while  $\mu_{\text{ns}}$  controls the amplitude of this force. When  $\sigma_{\text{ns}}$  goes to 0 and  $\mu_{\text{ns}}$  goes to  $\infty$  we arrive at a no slip condition.

Similarly, we desire to dampen bouncing. This is done with the viscous force

$$F_{\text{db}}(q_i, \dot{q}_i) = -\mu_{\text{db}} \dot{q}_i \left[ \frac{1}{2} \tanh \left( \frac{y_i}{\sigma_{\text{db}}} \right) - \frac{1}{2} \right] dy_i.$$

Finally, we incorporate gravity via the potential energy on each particle given by  $U_{\text{gr}}(x, y) = y$ . This potential energy imposes the force  $-dU_{\text{gr}}(x, y) = (0, -1)$  on each particle.

**2.3. The full model.** Now that we have established the Lagrangian of the walker, as well as the environmental forces imposed on it, we may finally provide the equations of motion. These equations of motion are obtained by adding the viscous forces,  $F_{\text{ns}}$  and  $F_{\text{db}}$ , and the potential forces,  $dU_{\text{np}}$  and  $dU_{\text{gr}}$ , to the equations for the walker in a vacuum. Therefore, the Lagrange-d'Alembert equations are given by

$$\begin{aligned} \ddot{q}_i = & \left( \sum_{j=1}^3 F_{ij}(\dot{q}_i, \dot{q}_j) \right) - \left( \sum_{i \neq j} \frac{\partial U_{ij}}{\partial q_i}(q_i, q_j) \right) \\ & - dU_{\text{np}}(q_i) + F_{\text{ns}}(q_i, \dot{q}_i) + F_{\text{db}}(q_i, \dot{q}_i) - dU_{\text{gr}}(q_i). \end{aligned}$$

We may express these equations in a more familiar form by defining the total force

$$F(q, \dot{q}) = \sum_{ij} F_{ij}(\dot{q}_i, \dot{q}_j) + \sum_i F_{\text{ns}}(q_i, \dot{q}_i) + F_{\text{db}}(q_i, \dot{q}_i)$$

and the total potential energy

$$U(q) = \sum_{i < j} U_{ij}(q_i, q_j) + \sum_i U_{\text{np}}(q_i) + U_{\text{gr}}(q_i)$$

to write the equations as

$$\ddot{q} = F(q, \dot{q}) - dU(q)$$

### 3. QUALITATIVE THEORY

**3.1. Reduction by Translation Symmetry.** Consider the left action of  $\mathbb{R}$  on  $\mathbb{R}^2$  given by translating the  $x$ -coordinate of each particle. We will denote this action by  $\rho : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . In this section we will show that our system is invariant with respect to this Lie group action. In particular, the action on  $Q$  which we also denote by  $\rho : \mathbb{R} \times Q \rightarrow Q$  is given by

$$\rho(s, (q_1, q_2, q_3)) := (\rho(s, q_1), \rho(s, q_2), \rho(s, q_3))$$

This action on  $Q$  can be lifted to  $TQ$  by the tangent lift. In coordinates, this is given by the map

$$\rho(s) \cdot (q, \dot{q}) \mapsto (\rho(s) \cdot q, \dot{q}).$$

It can be observed that  $L$  is invariant with respect to this action on  $TQ$ ; this is expressed formally by the observation

$$L(\rho(s) \cdot (q, \dot{q})) = L(q, \dot{q})$$

for any  $(q, \dot{q}) \in TQ$  and  $s \in \mathbb{R}$ . Additionally, the force  $F$  is really a map from  $TQ$  to  $T^*Q$  and is also  $\mathbb{R}$  invariant in the sense that

$$\langle F(\rho(s) \cdot (q, v_1)), \rho(s) \cdot (q, v_2) \rangle = \langle F(q, v_1), (q, v_2) \rangle$$

for any  $(q, v_1), (q, v_2) \in TQ$  and any  $s \in \mathbb{R}$ . Given that the equations of motion are written in terms of  $L$  and  $F$ , there must exist an equation of motion on the quotient space  $TQ/\mathbb{R}$ . The main goal of this section is to find these equations of motion on  $TQ/\mathbb{R}$ .

Let us begin by noting that  $\mathbb{R}$  acts by shifting the  $x$  coordinate and so  $Q/\mathbb{R} \equiv \mathbb{R}^5$ . We can coordinatize  $Q/\mathbb{R}$  by dropping the  $x$  coordinates of the third mass (this is one choice of many coordinate systems). This gives us coordinates  $(\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, \hat{y}_3)$  on  $Q/\mathbb{R}$  and induces the coordinates for  $TQ/\mathbb{R} \equiv \mathbb{R}^{11}$  given by  $(\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, \hat{y}_3, \hat{v}_1, \hat{w}_1, \hat{v}_2, \hat{w}_2, \hat{v}_3, \hat{w}_3)$ . We may construct the principal bundle projection  $\pi : TQ \rightarrow TQ/\mathbb{R}$ , which makes the assignment

$$\begin{aligned} \hat{x}_1 &= x_1 - x_3, & \hat{y}_1 &= y_1, & \hat{v}_1 &= v_1 - v_3, & \hat{w}_1 &= w_1 \\ \hat{x}_2 &= x_2 - x_3, & \hat{y}_2 &= y_2, & \hat{v}_2 &= v_2 - v_3, & \hat{w}_2 &= w_2 \\ & & \hat{y}_3 &= y_3, & \hat{v}_3 &= v_3, & \hat{w}_3 &= w_3 \end{aligned}$$

The equations of motion on  $TQ/\mathbb{R}$  can be written explicitly in terms of these coordinates. We find that the  $\mathbb{R}$ -invariant potential energies  $U_{ij}, U_{\text{np}} : Q \rightarrow \mathbb{R}$  may be written as functions  $\hat{U}_{ij}, \hat{U}_{f,i} : Q/\mathbb{R} \rightarrow \mathbb{R}$ , or more explicitly as

$$\begin{aligned} \hat{U}_{13}(\hat{q}) &= \frac{1}{2} \left( \sqrt{\hat{x}_1^2 + (\hat{y}_3 - \hat{y}_1)^2} - l_{13}(t) \right)^2 \\ \hat{U}_{23}(\hat{q}) &= \frac{1}{2} \left( \sqrt{\hat{x}_2^2 + (\hat{y}_3 - \hat{y}_2)^2} - l_{23}(t) \right)^2 \\ \hat{U}_{12}(\hat{q}) &= \frac{1}{2} \left( \sqrt{(\hat{x}_1 - \hat{x}_2)^2 + (\hat{y}_1 - \hat{y}_2)^2} - l_{12}(t) \right) \\ \hat{U}_{\text{np}}(\hat{q}_i) &= -\mu_{\text{np}} \tanh \left( \frac{\hat{y}_i}{\sigma_{\text{np}}} \right). \end{aligned}$$

We may define the total potential energy  $\hat{U} = \left( \sum_{i < j} \hat{U}_{ij} \right) + \sum_{i=1}^3 \hat{U}_{\text{np}}(\hat{q}_i)$ , which is simply the manifestation of  $U$  on the reduced space  $Q/\mathbb{R}$ .

Moreover, the forces are also  $\mathbb{R}$ -invariant. That is, for each force  $f : TQ \rightarrow T^*Q$  introduced in the previous section, there is a map  $\hat{f} : TQ/\mathbb{R} \rightarrow (TQ/\mathbb{R})^*$  characterized by the property

$$\langle \hat{f}(\pi(q, v)), \pi(q, \tilde{v}) \rangle = \langle f(q, v), (q, \tilde{v}) \rangle \quad \forall (q, v), (q, \tilde{v}) \in TQ.$$

Moreover, we may write the forces in terms of our coordinates on  $TQ/\mathbb{R}$ . In particular, they are:

$$\begin{aligned}
\hat{F}_{31} &= -\hat{v}_1(d\hat{x}_1 + dx_3) + (\hat{w}_3 - \hat{w}_1)d\hat{y}_1 \\
\hat{F}_{13} &= \hat{v}_1dx_3 + (\hat{w}_1 - \hat{w}_3)d\hat{y}_3 \\
\hat{F}_{32} &= -\hat{v}_2(d\hat{x}_2 + dx_3) + (\hat{w}_3 - \hat{w}_2)d\hat{y}_2 \\
\hat{F}_{23} &= \hat{v}_2dx_3 + (\hat{w}_2 - \hat{w}_3)d\hat{y}_3 \\
\hat{F}_{12} &= (\hat{v}_1 - \hat{v}_2)(d\hat{x}_2 + dx_3) + (\hat{w}_1 - \hat{w}_2)d\hat{y}_2 \\
\hat{F}_{21} &= (\hat{v}_2 - \hat{v}_1)(d\hat{x}_1 + dx_3) + (\hat{w}_2 - \hat{w}_1)d\hat{y}_1 \\
\hat{F}_{\text{ns},1} &= -\mu_{\text{ns}}(\hat{v}_1 + v_3) \left( \frac{1}{2} \tanh \left( \frac{\hat{y}_1}{\sigma_{\text{ns}}} \right) - \frac{1}{2} \right) (d\hat{x}_1 + dx_3) \\
\hat{F}_{\text{ns},2} &= -\mu_{\text{ns}}(\hat{v}_2 + v_3) \left( \frac{1}{2} \tanh \left( \frac{\hat{y}_2}{\sigma_{\text{ns}}} \right) - \frac{1}{2} \right) (d\hat{x}_2 + dx_3) \\
\hat{F}_{\text{ns},3} &= -\mu_{\text{ns}}\hat{v}_3 \left( \frac{1}{2} \tanh \left( \frac{\hat{y}_3}{\sigma_{\text{ns}}} \right) - \frac{1}{2} \right) dx_3 \\
\hat{F}_{\text{db},i} &= -\mu_{b,i}\hat{w}_i \left( \frac{1}{2} \tanh \left( \frac{\hat{y}_i}{\sigma_{\text{ns}}} \right) - \frac{1}{2} \right) d\hat{y}_i
\end{aligned}$$

where  $d\hat{x}_1, d\hat{x}_2, dx_3, d\hat{y}_1, d\hat{y}_2, d\hat{y}_3$  are dual vectors to the space spanned by the coordinates  $\hat{v}_1, \hat{v}_3, \hat{v}_3, \hat{w}_1, \hat{w}_2, \hat{w}_3$ . The dual vector  $dx_3$  is *not* the differential of the  $x_3$  coordinate because there is no  $x_3$  coordinate on the space  $Q/\mathbb{R}$ . The sequence of symbols “ $dx_3$ ” represents a dual vector to the space spanned by the coordinate  $\hat{v}_3$ . That  $dx_3$  is not the differential of a coordinate is related to the fact that  $\hat{v}_3$  is not associated to a base point and therefore is not a tangent vector in a traditional sense. This is because  $TQ/\mathbb{R}$  is not actually a tangent bundle, but is a generalization of a tangent bundle known as a Lie algebroid [13].

We find that the base coordinates on  $Q/\mathbb{R}$  evolve by

$$\begin{aligned}
\frac{d\hat{x}_1}{dt} &= \hat{v}_1 & , & & \frac{d\hat{y}_1}{dt} &= \hat{w}_1 \\
\frac{d\hat{x}_2}{dt} &= \hat{v}_2 & , & & \frac{d\hat{y}_2}{dt} &= \hat{w}_2 \\
& & & & \frac{d\hat{y}_3}{dt} &= \hat{w}_3
\end{aligned}$$

If we define the total force to be  $\hat{F}$ , then the evolution of  $\hat{v} = (\hat{v}_1, \hat{w}_1, \hat{v}_2, \hat{w}_2, \hat{v}_3, \hat{w}_3)$  is given by

$$\frac{d\hat{v}}{dt} = \hat{F} - d\hat{U}.$$



**3.2. Hyperbolic Stable Points.** In this section we will use the energy function on  $TQ/\mathbb{R}$  as a Lyapunov function to assert the existence of a hyperbolically stable point for the passive dynamics. In particular, consider the energy function

$$\hat{E}(\hat{q}, \hat{v}) = \frac{1}{2} \|\hat{v}\|^2 + \hat{U}(\hat{q}).$$

We can use the equations of motion to calculate the time derivative of  $\hat{E}$  along a trajectory

$$\frac{d\hat{E}}{dt} = \frac{\partial \hat{E}}{\partial \hat{q}} \frac{d\hat{q}}{dt} + \frac{\partial \hat{E}}{\partial \hat{v}} \frac{d\hat{v}}{dt} = d\hat{U}(\hat{q}) \cdot \hat{v} + \hat{v} \cdot (-d\hat{U}(\hat{q}) + \hat{F}(\hat{q}, \hat{v})) = \hat{F}(\hat{q}, \hat{v}) \cdot \hat{v}$$

It is simple to observe that  $\frac{d\hat{E}}{dt}$  is negative definite in  $\hat{v}$  since for any  $(q, v)$  such that  $(\hat{q}, \hat{v}) = \pi(q, v)$ , we find that

$$\hat{F}(\hat{q}, \hat{v}) \cdot \hat{v} = F(q, v) \cdot (q, v) = -\|v\|_*^2 + \sum_{i < j} v_i v_j + w_i w_j$$

where

$$\begin{aligned} \|v\|_*^2 &= \|v\|^2 + \sum_{i=1}^3 f_{\text{ns}}(y_i) v_i^2 + f_{\text{db}}(y_i) w_i^2 \\ f_{\text{ns}}(y) &= \mu_{\text{ns}} \tanh\left(\frac{y}{\sigma_{\text{ns}}}\right) \\ f_{\text{db}}(y) &= \mu_{\text{db}} \tanh\left(\frac{y}{\sigma_{\text{db}}}\right). \end{aligned}$$

Noting that  $\sum_{i < j} v_i v_j + w_i w_j \leq \|v\|^2$  we see that

$$\frac{d\hat{E}}{dt} \leq - \left( \sum_{i=1}^3 f_{\text{ns}}(y_i) v_i^2 + f_{\text{db}}(y_i) w_i^2 \right)$$

which is negative definite in  $\hat{v}$  and  $\hat{w}$  as long as  $y_i > 0$  for  $i = 1, 2, 3$ . One can verify that  $\hat{U}$  has isolated minima in  $Q/\mathbb{R}$  when  $\mu_{\text{np}}$  is sufficiently large. Moreover, these minima occur in the region  $y_i > 0$  for  $i = 1, 2, 3$ . If we denote one of these minima by  $\hat{q}_{\text{min}}$ , then  $\hat{q}_{\text{min}}$  is an asymptotically stable fixed point because the energy is decreasing monotonically until  $\hat{v} = \hat{w} = 0$ , at which point the state of the system must be at a critical point of  $\hat{U}$ .

**3.3. Time periodic perturbations.** Given a dynamical system  $\dot{x} = f(x)$  on a manifold  $M$  with a hyperbolically stable fixed point  $x_0$  in  $M$ , one can embed the system into a time-periodic augmented phase space  $S^1 \times M$  by using the vector field  $(\dot{\theta}, \dot{x}) = (1, f(x))$ . Then the orbit  $\gamma_0(t) = \{(\theta, x_0) : \theta \in S^1\}$  is a hyperbolically stable limit cycle for the system on  $S^1 \times M$ . Moreover, given a time-periodic perturbation  $f \mapsto f + \epsilon g_\theta$ , the limit cycle will continue to exist but be transformed into another limit cycle  $\gamma_\epsilon$  [4], [9] (see also “The Averaging Theorem” in [8]). A

cartoon of this idea is depicted in figure 3. We will abuse notation by referring to both the limit cycle on  $S^1 \times M$  and the projected cycle on  $M$  by the symbol  $\gamma$ .

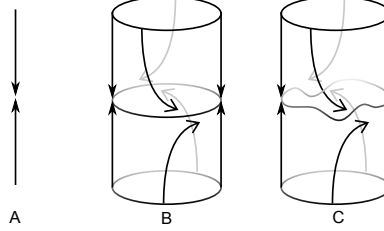


FIGURE 3. (A) A one dimensional system with a hyperbolic stable point. The embedding of the system in (A) into a time-periodic augmented phase space results in a limit cycle as shown in (B). Finally, a limit cycle will persist under a time-periodic perturbation as a diffeomorphism of the original limit cycle. This is drawn in (C).

In the previous subsection, we found a hyperbolically stable fixed point on  $TQ/\mathbb{R}$ . In this section, we will perturb this system by allowing the lengths  $\ell_{ij}$  to be time periodic with period  $T$ . If these oscillations are small, we can expect to observe a stable limit cycle,  $\gamma(t)$ , also with period  $T$  in the augmented phase space  $TQ/\mathbb{R}$ . However, this limit cycle is only a cycle on  $TQ/\mathbb{R}$ . In general, if  $\Gamma(t)$  is a trajectory in  $TQ$  such that  $\gamma(t) = \pi(\Gamma(t))$ , then it is generally not the case that  $\Gamma(t)$  is periodic. In particular, any periodic trajectory  $\gamma \subset TQ/\mathbb{R}$  of period  $T$  is reduced from a trajectory  $(q, \dot{q})(t) \in TQ$  such that

$$(q, v)(t + T) = \rho(\Delta x) \cdot (q, v)(t)$$

where  $\Delta x \in \mathbb{R}$  is obtained from the  $\hat{v}$  component of  $\gamma(t)$  via

$$\Delta x = \int_0^T \hat{v}_3(\tau) d\tau.$$

The above integral may be viewed as a phase reconstruction formula with respect to the reduction by  $\mathbb{R}$  symmetry. This relationship between trajectories in  $TQ$  and loops in  $TQ/\mathbb{R}$  is related to walking in the sense that the periodicity of  $\gamma(t) \in TQ/\mathbb{R}$  implies that  $(q, v)(t) \in TQ$  satisfies

$$(q, v)(t + kT) = \rho(k\Delta x) \cdot (q, v)(t).$$

Simply put, the motion in  $TQ$  is *relatively periodic* in that each period is related to the previous period by a shift to the right by an amount  $\Delta x$ . For this reason, a trajectory  $(q, v)(t) \in TQ$  is called a *relative limit cycle* if when  $\gamma(t) = \pi((q, v)(t)) \in TQ/\mathbb{R}$  is a limit cycle. Moreover, this behavior in  $TQ$  is stable because the limit cycle  $\gamma(t) \in TQ/\mathbb{R}$  is stable. Such behavior could resemble walking which is characterized by being relatively periodic and stable.

**3.4. Early Numerical Results.** I hypothesize that the shift  $\Delta x$  is generally non-zero. We can provide evidence in support of this hypothesis through the help of numerics. In this section, we use the stiff ODE solver of the *python scientific computing library* with an absolute tolerance of 1e-10 to integrate the equations of motion. With the parameters

$$\begin{aligned}\mu_{\text{ns}} &\mapsto 10.0 & \sigma_{\text{ns}} &\mapsto 0.1 \\ \mu_{\text{db}} &\mapsto 1.00 & \sigma_{\text{db}} &\mapsto 0.1 \\ \mu_{\text{np}} &\mapsto 10.0 & \sigma_{\text{np}} &\mapsto 0.1\end{aligned}$$

along with the time-dependent lengths

$$\begin{aligned}l_{13} &= 1 - 0.3 \sin \left( \omega \left( t - \frac{3}{2} \right) \right) \\ l_{13} &= 1 - 0.3 \sin (\omega t) \\ l_{23} &= 3 - l_{12} - l_{13},\end{aligned}$$

where the frequency  $\omega = 2\pi$ . This particular choice of parameters yields behavior which suggests that we are observing a relative limit cycle. A typical trajectory is depicted in figure 4 and the motion of the masses in a reference frame attached to the center of the body is depicted in figure 5.

We observe an initial transient period followed by orderly periodic behavior. Over large times, this results in a constant drift of the walker towards decreasing values of  $x$ , while the  $y$ -coordinate of the walker stays just above  $y = 0$ . Figure 4 also provides a zoomed in view to illustrate what the evolution looks like on smaller time scales. On these smaller time scales we can see that the  $x$ -coordinate oscillates with an angular frequency of  $2\pi$  and a gradual but steady drift towards decreasing  $x$ , while the  $y$ -coordinate tends to oscillate with an angular frequency of  $2\pi$  without any drift at all. To further verify this behavior, we have also provided a plot of the trajectories in a moving reference frame such that the average  $x$  coordinate of the masses is 0 in figure 5. One can interpret figure 5 as a plot of the trajectories of the masses in  $Q/\mathbb{R}$ . One can observe some initial transient dynamics before these masses settle into cyclic behavior. The long-time cyclic behavior manifests as three thick circles in figure 5 located roughly at the coordinates  $(-0.5, 0.3)$ ,  $(0.0, 1.0)$ , and  $(0.5, 0.3)$ .

Altering parameter values and the initial conditions always produces behavior indicative of a relative limit cycle. However, the numerical results should be taken with a grain of salt, because multiple hyperbolic stable equilibria exist in  $Q/\mathbb{R}$ . In particular, given an equilibrium in  $Q/\mathbb{R}$  one can permute the masses with one another to achieve another equilibrium. For example, the evolution depicted in figure 4 suggests that the system almost settles near one of these equilibrium configurations before the yellow mass and the red mass appear to swap positions between  $t = 10$  and  $t = 20$ . One could imagine varying the initial condition only slightly, in such a way that yellow and the red mass do not sway and the system

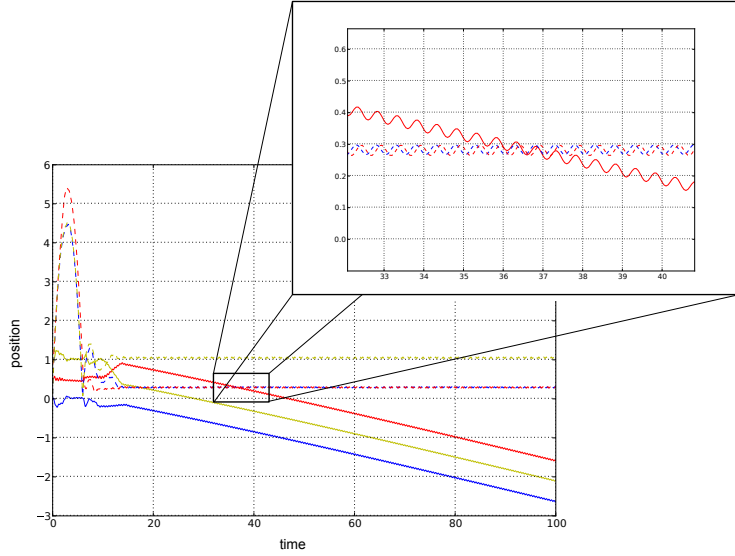


FIGURE 4. This plot depicts the evolution of a typical trajectory. The solid lines correspond to the  $x$  coordinates of the three masses, while the dotted lines correspond to the  $y$  coordinates. We can clearly see a constant drift of the  $x$ -coordinate of each mass. If we blow-up a region we can directly observe behavior which suggests the system is captured in a stable periodic orbit in  $TQ/\mathbb{R}$ .

settles to a different relative limit cycle. This is just one particular example. Each equilibrium can be periodically perturbed to produce a limit cycle, and it is generally not the case that the same walking behavior will result for different limit cycles. A natural extension of this paper is understanding the consequences of having multiple hyperbolically stable equilibria.

Secondly, while the trajectories appear to have converged to a limit cycle, it is not clear that absolutely all of the transient dynamics have left the system. There may be some slowly varying dynamics which have not dissipated. Further analytics may confirm or reject this possibility. Nonetheless, it is apparent that limit cycle-like behavior dominates fairly quickly in these systems.

#### 4. CONCLUSION & FUTURE WORK

In this paper we have presented a model of walking where the walker consists of three point particles in  $\mathbb{R}^2$  connected by springs, and the ground consists of the sum of a smooth potential energy and a smooth viscous force on the phase space,  $TQ$ . We then found that the system exhibits a symmetry with respect to translation of the  $x$ -coordinate. The reduced equations of motion on  $TQ/\mathbb{R}$  were dissipative, and lead to hyperbolically stable equilibria. By periodically perturbing the system we

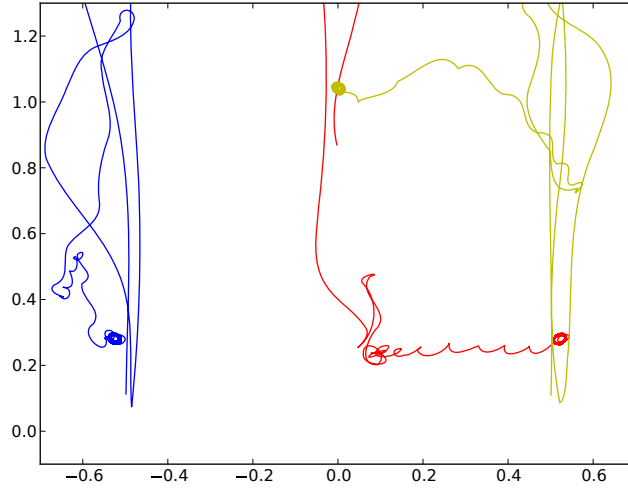


FIGURE 5. This plot depicts the location of the three masses in a reference frame where the average  $x$ -coordinate of the masses has been translated to the origin. Therefore this plot depicts the trajectory of the masses in  $Q/\mathbb{R}$ . Here we can observe limit cycle behavior in that each of the masses settles to stable oscillatory behavior after an initial period of transient dynamics.

obtained limit cycles in  $TQ/\mathbb{R}$  in the vicinity of the equilibria. When these limit cycles in  $TQ/\mathbb{R}$  were lifted to relative limit-cycles in  $TQ$  we observed dynamics which resembled walking. Numerical simulations then provided further evidence for this interpretation. The work done so far suggests a number of avenues to pursue. In particular:

- (1) While we observe a limit cycle for the regularized model, it is not clear that this limit cycle persists in the limit as  $\sigma_{\text{ns}}, \sigma_{\text{np}} \rightarrow 0$ . Proving that this is (or is not) the case can be approached by viewing this as a singular perturbation problem.
- (2) As previously mentioned, while the numerics may make one initially happy, it is not clear that all of the transient dynamics have left the system. The influence of time-scales on this system is particularly important as the problem is likely exacerbated as the width of the regularization shrinks to zero. Deriving lower bounds on the decay rates for the transient dynamics could help resolve this issue.
- (3) While the limit cycle in the paper is stable, the size of the stability basin is not mentioned. Having a large stability basin is one method of achieving robustness and so having a lower bound for the radius if this basin would be useful.

In summary, it appears that regularized models are capable of exhibiting behavior which resembles walking. Such models are open to classical techniques in dynamical systems, and allow one to view walking as a limit cycle in a reduced space in which case the absolute motion manifests as a phase shift. We hope that the simplicity of this characterization encourages further research down this avenue which can enhance observations as well as complement the hybrid systems approach.

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